Extensions of Rokhlin congruence for curves on surfaces

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Abstract

The subject of this paper is the problem of arrangement of a real nonsingular algebraic curve on a real non-singular algebraic surface. This paper contains new restrictions on this arrangement extending Rokhlin and Kharlamov-Gudkov-Krakhnov congruences for curves on surfaces.

1 Introduction

If we fix a degree of a real nonsingular algebraic surface then in accordance with Smith theory there is an upper bound for the total $\mathbf{Z}_2$-Betti number of this surface (the same thing applies to curves). The most interesting case according to D.Hilbert is the case when the upper bound is reached (in this case the surface is called an M-surface).

Let A be a real nonsingular algebraic projective curve on a real nonsingular algebraic projective surface B. If A is of even degree in B then A divides B into two parts B_+ and B_- (corresponding to areas of B where polynomial determining A is positive and negative).

Rokhlin congruence [1] yields a congruence modulo 8 for Euler characteristic χ of B_+ provided that

(i) B is an M-surface

(ii) A is an M-curve

(iii) B_+ lies wholly in one component of B

(iv) $rk(in_* : H_1(B_+; \mathbf{Z}_2) \to H_1(B; \mathbf{Z}_2)) = 0$

(v) if the degree of polynomial determining A in B is congruent to 2 modulo 4 then all components of B containing no components of A are contractible in P^q [1]

Kharlamov-Gudkov-Krakhnov [2],[3] congruence yields congruence modulo 8 for $\chi(B_+)$ under assumptions (i),(iii),(v) and either assumption that A is an (M-1)-curve and $rk(in_*) = 0$ or assumption that A is an M-curve and $rk(in_*) = 1$ and all components of A are $\mathbf{Z}_2$-homologically trivial. Recall that Rokhlin congruence for surfaces yields

[1] *Remark*. Paper [1] contains a mistake in the calculation of characteristic class of covering $Y \to \mathbf{C}B$. It leads to the omission (after reformulation) of (v) in assumptions of congruence. The proof given in [1] really uses (v).

congruence modulo 16 for $\chi(B)$ so a congruence modulo 8 for $\chi(B_+)$ is equivalent to a congruence modulo 8 for $\chi(B_-)$.

One of the properties of M-surfaces (similar to the property of M-curves) remarked by V.I.Arnold [4] is that a real M-surface is a characteristic surface in its own complexification (for M-curves it means that a real M-curve divides its own complexification since a complex curve is orientable).We shall say that a real surface is of a characteristic type if it is a characteristic surface in its own complexification.Note that the notion of characteristic type of surfaces is analogous to the notion of type I of curves.

Consider at first the weakening of assumption (i) in Rokhlin and Kharlamov-Gudkov-Krakhnov congruences.Instead of (i) we can only assume that B is of characteristic type.

According to O.Ya.Viro [5] there are some extra structures on real surfaces of characteristic type ,namely, Pin_--structures and semiorientations or relative semiorientations (semiorientation is the orientation up to the reversing). In this paper we introduce another structure on surfaces of characteristic type — complex separation which is also determined by the arrangement of a real surface in its complexification. The complex separation is a natural separation of the set of components of a real surface of characteristic type into two subsets.Note that the set of semiorientations of a surface is an affine space over $\mathbf{Z}_2$-vector space of separations of this surface.

We use the complex separation to weaken assumption (iii). In theorem 1 instead of (iii) we assume only that B_+ lies in components of one class of complex separation.

The further extension ,theorem 2, can be applied not only for curves of even degree but also sometimes for curves of odd degree. Another weakening of assumptions in theorem 2 is that components of a curve are not necessarily $\mathbf{Z}_2$-homologically trivial.

These extensions can be applied to curves on quadrics and cubics. Theorem 1 together with an analogue of Arnold inequality for curves on cubics gives a complete system of restrictions for real schemes of flexible curves of degree 2 on cubics of characteristic type (see [6]).An application of theorem 2 to curves on an ellipsoid gives a complete system of restrictions for real schemes of flexible curves of degree 3 on an ellipsoid and reduces the problem of classification of real schemes of flexible curves of degree 5 on an ellipsoid to the problem of the existence of two real schemes (see [7]). An application of theorem 2 to curves on a hyperboloid extends Matsuoka congruences [8] for curves with odd branches on a hyperboloid (see [9]).

Applications of theorem 2 to empty curves on surfaces give restrictions for surfaces involving complex separation of surfaces. Restrictions for curves of even degree on surfaces can be obtained also by the application of these restrictions for surfaces to the 2-sheeted covering of surface branched along the curve , if we know the complex separation of this covering. This complex separation is determined by complex orientation of the curve.For example in this way one can obtain new congruences for complex orientations of curves on a hyperboloid.

These applications and further extension of Rokhlin congruence for curves on surfaces will be published separately in [9]. For example the assumption that the surface and the curve on the surface are complete intersections is quite unnecessary ,but this assumption simplifies definition of number c in formulations of theorems. The formulations of these results were announced in [7] as well as the formulations of results of the present paper.

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2 Notations and formulations of main theorems

Let the surface B be the transversal intersection of hypersurface in P^q defined by equations $P_j(x_0,\dots,x_q)=0, j=1,\dots,s-1$; $\mathbf{C}B$ and $\mathbf{R}B$ be sets of complex and real points of B; let A be a nonsingular curve on B defined by an equation $P_s(x_0,\dots,x_q)=0$,where P_j is a real homogeneous polynomial of degree m_j $j=1,\dots,s$, $q=s+1$; $\mathbf{C}A$ and $\mathbf{R}A$ be sets of complex and real points of A. Let $conj$ denote the involution of complex conjugation. Set c to be equal to $\frac{\prod_{j=1}^{s-1} m_j}{4}$.If m_s is even then denote $\{x\in\mathbf{R}B|\pm P_s(x)\geq 0\}$ by $B_\pm$ and set d to be equal to $rk(in_*: H_1(B_+;\mathbf{Z}_2)\to H_1(\mathbf{R}B;\mathbf{Z}_2))$.

A real algebraic variety is called an (M-j)-variety if its total $\mathbf{Z}_2$-Betti number is less by $2j$ then total $\mathbf{Z}_2$-Betti number of its complexification (Harnack-Smith inequality shows that $j\geq 0$). Let A be an (M-k)-curve. Let D_M be the operator of Poincaré duality of manifold M.We shall say that B is a surface of characteristic type if $[\mathbf{R}B]=D_{\mathbf{C}B}w_2(\mathbf{C}B)\in H_2(\mathbf{C}B;\mathbf{Z}_2)$ (as it is usual we denote by $[\mathbf{R}B]$ the element of $H_2(\mathbf{C}B;\mathbf{Z}_2)$ realized by $\mathbf{R}B$). We shall say that (B,A) is a pair of characteristic type if $[\mathbf{R}B]+[\mathbf{C}A]+D_{\mathbf{C}B}(w_2(\mathbf{C}B))=0\in H_2(\mathbf{C}B;\mathbf{Z}_2)$.It is said that A is a curve of type I if $[\mathbf{R}A]=0\in H_1(\mathbf{C}A;\mathbf{Z}_2)$.It is said that A is of even(odd) degree if $[\mathbf{C}A]=0\in H_2(\mathbf{C}B;\mathbf{Z}_2)$ (otherwise).

As it is usual we denote by σ and χ the signature and the Euler characteristic.By $\beta(q)$ we mean Brown invariant of $\mathbf{Z}_4$-valued quadratic form q.

Theorem 1 *If B is a surface of characteristic type then there is defined a natural separation of surface $\mathbf{R}B$ into two closed surfaces B_1 and B_2 by the condition that $B_j, j=1,2$, is a characteristic surface in $\mathbf{C}B/conj$.Suppose that m_s is even,$B_+\subset B_1$, every component of $\mathbf{R}A$ is $\mathbf{Z}_2$-homologous to zero in $\mathbf{R}B$ and if $m_s\equiv 2 \pmod 4$ then suppose besides that B_2 is contractible in $\mathbf{R}P^q$.*

- **a)** *If $d+k=0$ then $\chi(B_+)\equiv c \pmod 8$.*
- **b)** *If $d+k=1$ then $\chi(B_+)\equiv c\pm 1 \pmod 8$.*
- **c)** *If $d+k=2$ and $\chi(B_+)\equiv c+4 \pmod 8$ then A is of type I and B_+ is orientable.*
- **d)** *If A is of type I and B_+ is orientable then $\chi(B_+)\equiv c \pmod 4$.*

Theorem 2 *If (B,A) is a pair of characteristic type then there is defined a natural separation of surface $\mathbf{R}B\setminus\mathbf{R}A$ into two surfaces B_1 and B_2 with common boundary $\mathbf{R}A$ by the condition that $B_j\cup\mathbf{C}A/conj$ is a characteristic surface in $\mathbf{C}B/conj$, there is defined Guillou-Marin form q_j on $H_1(B_j\cup\mathbf{C}A/conj;\mathbf{Z}_2)$ and*

$$\chi(B_j)\equiv c+\frac{\chi(\mathbf{R}B)-\sigma(\mathbf{C}B)}{4}+\beta(q_j) \pmod 8.$$

3 Proof of theorem 2

Consider the Smith exact sequence of double branched covering $\pi:\mathbf{C}B\to\mathbf{C}B/conj$

$$\xrightarrow{\beta_3} H_3(\mathbf{C}B/conj,\mathbf{R}B;\mathbf{Z}_2)\xrightarrow{\gamma_3} H_2(\mathbf{R}B;\mathbf{Z}_2)\oplus H_2(\mathbf{C}B/conj,\mathbf{R}B;\mathbf{Z}_2)\xrightarrow{\alpha_2} H_2(\mathbf{C}B;\mathbf{Z}_2)\xrightarrow{\beta_2}.$$

Let ϕ denote the composite homomorphism

$$H_2(\mathbf{C}B/conj, \mathbf{R}B; \mathbf{Z}_2) \stackrel{0\oplus id}{\to} H_2(\mathbf{R}B; \mathbf{Z}_2) \oplus H_2(\mathbf{C}B/conj, \mathbf{R}B; \mathbf{Z}_2) \stackrel{\alpha_2}{\to} H_2(\mathbf{C}B; \mathbf{Z}_2).$$

Let j denote the inclusion map $(\mathbf{C}B/conj, \emptyset) \to (\mathbf{C}B/conj, \mathbf{R}B)$.Recall that $\phi_* \circ j_*$ is equal to Hopf homomorphism $\pi^!$. It is easy to deduce from the exactness of the Smith sequence that ϕ is a monomorphism. Indeed, $\pi_1(\mathbf{C}B) = 0$ hence $\pi_1(\mathbf{C}B/conj) = 0$ and $H_3(\mathbf{C}B/conj; \mathbf{Z}_2) = 0$. Therefore boundary homomorphism $\partial : H_3(\mathbf{C}B/conj, \mathbf{R}B; \mathbf{Z}_2) \to H_2(\mathbf{R}B; \mathbf{Z}_2)$ is a monomorphism.Therefore,since ∂ is the first component of γ_3, $Im\gamma_3 \cap (\{0\} \oplus H_2(\mathbf{C}B/conj, \mathbf{R}B; \mathbf{Z}_2)) = 0$ and ϕ is a monomorphism.

It is easy to check that

$$\pi^* w_2(\mathbf{C}B/conj) = D^{-1}_{\mathbf{C}B}[\mathbf{R}A] + w_2(\mathbf{C}B).$$

Thus $\pi^!(D_{\mathbf{C}B/conj} w_2(\mathbf{C}B/conj)) = [\mathbf{C}A]$, therefore,because of the injectivity of ϕ,we obtain that

$$j_* D_{\mathbf{C}B/conj} w_2(\mathbf{C}B/conj) = [\mathbf{C}A/conj, \mathbf{R}A] \in H_2(\mathbf{C}B/conj, \mathbf{R}B; \mathbf{Z}_2).$$

It means that there exists a compact surface $B_1 \subset \mathbf{R}B$ with boundary $\mathbf{R}A$ such that $B_1 \cup \mathbf{C}A/conj$ is a characteristic surface in $\mathbf{C}B/conj$. Surface $\mathbf{R}A$ is homologous to zero in $\mathbf{C}B/conj$ since $\mathbf{R}A$ is the set of branch points of π. Set B_2 to be equal to the closure of $(\mathbf{R}B \setminus B_1)$. Then $B_2 \cup \mathbf{C}A/conj$ is a characteristic surface $\mathbf{C}B/conj$, $B_1 \cup B_2 = \mathbf{R}B$,$B_1 \cap B_2 = \partial B_1 = \partial B_2 = \mathbf{R}A$.

Let us prove the uniqueness of pair $\{B_1, B_2\}$.It is sufficient to prove that the dimension of the kernel of inclusion homomorphism $H_2(\mathbf{R}B; \mathbf{Z}_2) \to H_2(\mathbf{C}B/conj; \mathbf{Z}_2)$ is equal to 1. This follows from the equality $\dim H_3(\mathbf{C}B/conj, \mathbf{R}B; \mathbf{Z}_2) = 1$ that can be deduced from the exactness of the Smith sequence.

We apply now Guillou-Marin congruence [10] to pair $(\mathbf{C}B/conj, B_j \cup \mathbf{C}A/conj), j = 1, 2$

$$\sigma(\mathbf{C}B/conj) \equiv [B_j \cup \mathbf{C}A/conj] \circ [B_j \cup \mathbf{C}A/conj] + 2\beta(q_j) \pmod{16}.$$

Hirzebruch index theorem gives an equality $\sigma(\mathbf{C}B/conj) = \frac{\sigma(\mathbf{C}B) - \chi(\mathbf{R}B)}{2}$.To finish the proof note that $[B_j \cup \mathbf{C}A/conj] \circ [B_j \cup \mathbf{C}A/conj] = 2c - 2\chi(B_j)$ (the calculation is similar to Marin calculation in [11]).

4 Proof of the theorem 1

Pair $(B, \emptyset)$ is of characteristic type since A is of even degree in B. Thus the first part of theorem 1 follows from theorem 2 — there exist a natural separation of B into two surfaces B_1 and B_2 such that B_1 and B_2 are characteristic surfaces in $\mathbf{C}B/conj$.

Let V denote $B_+ \cup \mathbf{C}A/conj$.Let W denote $V \cup B_1$. Recall that $B_+ \cap B_2 = \emptyset$ thus $V \cap B_2 = \emptyset$.

Lemma 1 $[V] = 0 \in H_2(\mathbf{C}B/conj; \mathbf{Z}_2)$

Proof. Since A is of even degree in B, there exists a 2-sheeted covering $p : Y \to \mathbf{C}B$ branched along $\mathbf{C}A$. Involution $conj$ can be lifted to involutions T_+ and $T_- : Y \to Y$

since $\mathbf{C}A$ is invariant under $conj$. It is easy to see using the straight algebraic construction of p that T_+ and T_- can be chosen in such a way that the set of fixed points of $T_\pm$ is $p^{-1}(B_\pm)$.

Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d] & \mathbf{C}B \arrow[d, "\pi"] \\ Y/T_- & \mathbf{C}B/conj. \end{tikzcd}$$

This diagram can be expanded to a commutative one by map $p^{'} : Y/T_{\mp} \to \mathbf{C}B/conj$. It is easy to see that $p^{'}$ is a 2-sheeted covering branched along V. Therefore $[V] = 0 \in H_2(\mathbf{C}B/conj; \mathbf{Z}_2)$.

Using Lemma 1 we see that W is a characteristic surface in $\mathbf{C}B/conj$ as well as B_2.We apply Guillou-Marin congruence to these two surfaces :

$$\sigma(\mathbf{C}B/conj) \equiv [W] \circ [W] + 2\beta(q_W) \equiv 2c - 2\chi(B_+) - 2\chi(B_2) + 2\beta(q_W) \pmod{16}$$

$$\sigma(\mathbf{C}B/conj) \equiv [B_2] \circ [B_2] + 2\beta(q_{B_2}) \equiv -2\chi(B_2) + 2\beta(q_{B_2}) \pmod{16}$$

,where q_W and q_{B_2} are Guillou-Marin forms of W and B_2. Therefore

$$\chi(B_+) \equiv c + \beta(q_W) - \beta(q_{B_2}) \pmod{8}.$$

Lemma 2 $\forall x \in H_1(B_2; \mathbf{Z}_2), q_{B_2}(x) - q_W(x) = \begin{cases} 0 & \textit{if } x \textit{ is contractible in } \mathbf{R}P^q \\ \frac{m_s}{2} & \textit{if } x \textit{ is noncontractible in } \mathbf{R}P^q \end{cases}.$

Proof. It follows from the definition of Guillou-Marin form that values on x of q_{B_2} and q_W are differed by linking number of x and V in $\mathbf{C}B/conj$ that is equal to linking number of x and $\mathbf{C}A$ in $\mathbf{C}B$. The last linking number can be calculated from the straight construction of a 2-sheeted covering branched along $\mathbf{C}A$.

It was shown in [12] that Brown invariant of form q on the union of two surfaces with common boundary is equal to the sum of Brown invariants of restrictions of q on these surfaces in the case when q vanishes on the common boundary. Now theorem 1 follows from this additivity of Brown invariant and the classification of low-dimensional $\mathbf{Z}_4$-valued quadratic forms (see[12]). Indeed, since every component of $\mathbf{R}A$ is homologous to 0 in $\mathbf{R}B$, $\beta(q_W) = \beta(q_W|_{\mathbf{C}A/conj}) + \beta(q_W|_{B_+}) + \beta(q_W|_{B_2})$. Lemma 2 shows that under assumptions of theorem 1 $\beta(q_W|_{B_2}) = \beta(q_{B_2})$. To complete the proof note that ranks of intersection forms on $H_1(B_+; \mathbf{Z}_2)$ and $H_1(\mathbf{C}A/conj; \mathbf{Z}_2)$ are equal to d and k respectively.

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